JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 58, 98–112 (1977)

Rest Point Theorems for Autonomous Control Systems

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The classical Poincaré–Bendixson theory in the plane is generalized to multivalued vector fields. For control systems $\dot{x} = f(x, u)$ where the state x is in $\mathbf{R}^n$ and the control u is valued in a compact subset of $\mathbf{R}^m$ we study the existence (and nonexistence) of rest states. Some special emphasis is focused on the case $n = 2$.

1. INTRODUCTION

Consider the autonomous control system given by

$$\dot{x} = f(x, u) \tag{1.1}$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$ denote the state and control vectors, respectively. For a given set $\Omega \subset \mathbf{R}^m$ we will denote by U_Ω the set of *admissible controls*, i.e., the space of Lebesgue measurable functions u: $[0, \infty) \to \Omega$. For an admissible control $u \in U_\Omega$, a solution of (1.1) (if it exists) will be denoted by $\varphi_u(t)$, and if the solution satisfying the initial condition $x(0) = x_0$ is known to exist and to be unique on $[0, \infty)$ we shall denote it by $x(t) = \varphi(t, 0, x_0, u)$. In what follows it will be clear from the context whenever u represents a vector in Ω or a control in U_Ω.

* Supported in part by US Army Research Grant DAHC04-76-G-0011 and by US Air Force Grant 72-2268 through the Center for Mathematical System Theory, University of Florida, Gainesville, Florida 32611, USA.

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ISSN 0022-247X

By an *Ω-rest state* of system (1.1) is meant a vector $x \in \mathbf{R}^n$ such that $0 \in f(x, \Omega) := \{f(x, u) \mid u \in \Omega\}$, that is, a state at which system (1.1) can be held indefinitely by a constant control. For the case where system (1.1) is independent of u, i.e., when (1.1) describes a classical dynamical system, the concept of Ω-rest state reduces to that of a critical point.

The existence of rest points in control systems received recently some attention by Feuer and Heymann [4] and Heymann and Stern [5, 6]. Interest in this problem derives from its relevance to questions of controllability and stabilizability of control systems. Feuer and Heymann [4] proved that an Ω-rest state exists in a compact convex weakly Ω-invariant set; that is, a set X such that for each $x_0 \in X$ there exists an admissible control $u \in U_\Omega$ whose corresponding solution satisfies $\varphi(t, 0, x_0, u) \in X$ for all $t \geqslant 0$. This result generalizes to a control theoretic setting the classical theorem of dynamical systems which asserts that a compact homeomorphically convex positively invariant set contains a critical point (see, e.g., Bhatia and Szego [1]). In Heymann and Stern [5] various other results on the existence of Ω-rest states were obtained and in particular, it was proved that an Ω-rest state exists in a complementary weakly Ω-invariant compact convex set, i.e., a set X such that $\overline{X^c}$ (the closure of its complement) is weakly Ω-invariant.

In the present paper the investigation of existence of Ω-rest states and related problems is further expanded. In Section 2 the classical Poincaré–Bendixson theorem (which states that the trajectory of a periodic solution of an autonomous differential equation in the plane encloses a critical point) is generalized to multivalued vector fields. Specifically, the differential equation is replaced by a contingent equation of the form $\dot{x} \in V(x)$ where $V(x)$ is a compact convex set which depends continuously on x. The main result of this section is Theorem 2.16 which states that if a compact connected set contains a trajectory (absolutely continuous solution) of the contingent equation for all time t ($t \in [0, \infty)$) then its simply connected hull contains a critical point (i.e., a point x such that $0 \in V(x)$). In Section 3 the results of Section 2 are applied to planar control systems and it is shown that if a periodic solution does not pass through any Ω-rest states, then for each $\bar{u} \in \Omega$ there exists an $\bar{x}$ enclosed by the trajectory, such that $f(\bar{x}, \bar{u}) = 0$. It is also shown that under very mild conditions (convexity of the velocity sets) the existence of bounded controlled trajectories implies the existence of Ω-rest states in the plane, a result which does not generalize to systems in higher dimensional spaces (except in special cases).

For $x \in \mathbf{R}^n$ we define the *reachable set from* x in time $t > 0$ (for system (1.1)) as the set $F_t(x) := \{\varphi(t, 0, x, u) \mid u \in U_\Omega\}$. We shall say that the set $X \subset \mathbf{R}^n$ is *Ω-constrained* in time $t > 0$ provided that for all $x \in X$ we have

$$H(F_t(x)) \cap X \neq \varnothing \qquad \text{(nonempty intersection)}$$

where $H(\cdot)$ denotes the convex hull. A set X with nonempty interior will be called *strongly Ω-constrained* if there exists $t > 0$ such that for all $x \in \bar{X}$ (the closure of X) and all $\tau \geqslant t$

$$H(F_\tau(x)) \cap \operatorname{int}(X) \neq \varnothing$$

where int$(\cdot)$ denotes interior. A point x is called *t-hull periodic* (where $t > 0$) if $x \in H(F_t(x))$.

In Section 4 we prove a general result (Theorem 4.6) which states that if the boundary of an Ω-constrained compact convex set X with nonempty interior in $\mathbf{R}^n$ contains no hull periodic points then X must contain an Ω-rest state for each $u \in \Omega$. As an interesting corollary to Theorem 4.6 we obtain a result of Jones [7] (see also Jones and Yorke [8] and Remark 4.21 below) which states that, in dynamical systems, homeomorphically convex compactly constrained open subsets of $\mathbf{R}^n$ contain critical points. (An open set $X \subset \mathbf{R}^n$ is called *compactly constrained* with respect to a dynamical system if there exists a time $t^* > 0$ such that every point of $\bar{X}$ leads back into and remains in X for all $\tau \geqslant t^*$.) Finally we show via an example in Section 5 that for $n \geqslant 3$, strongly Ω-constrained compact convex subsets with nonempty interior do not necessarily contain Ω-rest states. Hence the control theoretic analog of the Jones–Yorke result fails to hold for $n \geqslant 3$, a fact which illustrates sharply the added complexity which the presence of control introduces.

2. The Poincaré–Bendixson Theorem for Multivalued Vector Fields

Let Γ denote the metric space of nonempty convex subsets of $\mathbf{R}^2$ with Hausdorff topology. For topological spaces X and Y denote by $C(X \to Y)$ the space of continuous mappings $f\colon X \to Y$. For a subset $S \subset \mathbf{R}^2$ let V be a (multivalued) *vector field* on S, i.e., $V \in C(S \to \Gamma)$. A point $x \in S$ is called a *critical* point of V if $0 \in V(x)$. The vector field V is called *regular* on S if there are no critical points of V on S.

Let K be a Jordan curve. For a real number $T > 0$, a mapping $x \in C([0, T] \to \mathbf{R}^2)$ is called a *proper parametrization* of K if the following conditions hold:

(2.1) $K = \{x(t) \mid 0 \leqslant t \leqslant T\}$,

(2.2) $x(t) \neq x(s)$ for all $0 \leqslant s < t < T$,

(2.3) $x(0) = x(T)$.

From the definition of a Jordan curve it follows that a proper parametrization always exists. If K is a Jordan curve and $v \in C(K \to \mathbf{R}^2)$ is a (single valued) vector field defined and regular on K, we denote by $\rho_K(v)$ the *index* (or *degree*) of v on K. (For definition and properties of the index see, e.g., Dugundji [3] or Lefschetz [10].) Finally, if K is a Jordan curve, we denote by enc(K) the bounded component of the complement of K and we define $\overline{\text{enc}}(K) = K \cup \text{enc}(K)$.

Let $D \subset \mathbf{R}^2$ be an open domain and let $V \in C(D \to \Gamma)$ be a multivalued vector field. Consider the contingent equation

$$\dot{x}(t) \in V(x(t)). \tag{2.4}$$

For $t > 0$ a function $\varphi \in C([0, T] \to D)$ is a solution of (2.4) if it is absolutely continuous and $\dot{\varphi}(t) \in V(\varphi(t))$ a.e. on $[0, T]$.

For a compact connected subset $M \subset \mathbf{R}^2$ define the *simply connected hull* of M, denoted $SCH(M)$, as the complement of the unbounded component of M^c (M^c denoting the complement of M). It is then easily verified that $SCH(M)$ is compact, simply connected. (Note, however, that even if M is compact and simply connected it is in general false that $SCH(M) = M$.)

We can now state the first principal results of this section.

(2.5) THEOREM. *Let V be a multivalued vector field defined on a domain $D \subset \mathbf{R}^2$. Let $T > 0$ and let φ be a solution of the contingent equation* (2.4) *on the interval* $[0, T]$ *such that* $\varphi(0) = \varphi(T)$. *Then $SCH(L)$ contains a critical point, where $L := \{\varphi(t) \mid 0 \leqslant t \leqslant T\}$ is the trajectory of φ.*

The proof of Theorem 2.5 depends on the following central

(2.6) THEOREM. *Let V be a multivalued vector field defined on a domain $D \subset \mathbf{R}^2$, and for some $T > 0$ let φ be a solution of the contingent equation* (2.4) *on the interval* $[0, T]$. *If φ is a proper parametrization of a Jordan curve K then $\overline{\mathrm{enc}}(K)$ contains a critical point of V.*

(2.7) *Remark.* In the classical setting of the Poincaré–Bendixson theorem the solutions of the differential equation are continuously differentiable. If we would consider only continuously differentiable solutions of (2.4) the generalization would be immediate. However, the fact that our solutions are much weaker in that they are only absolutely continuous functions, complicates matters a great deal.

For the proof of Theorems 2.5 and 2.6 we shall need several auxiliary results.

(2.8) LEMMA. *If $D \subset \mathbf{R}^2$ is open and $V \in C(D \to \Gamma)$, then V admits a* continuous selection, *i.e., there exists $v \in C(D \to \mathbf{R}^2)$ such that $v(x) \in V(x)$ for all $x \in D$.*

Lemma 2.8 is well known (in fact it holds in much more general spaces) and one continuous selection can be made by choosing for $v(x)$ the unique point of $V(x)$ with least Euclidean distance from x.

(2.9) LEMMA. *If v_1 and v_2 are any continuous selections of a vector field $V \in C(D \to \Gamma)$ and K is a Jordan curve such that V is regular on K, then $\rho_K(v_1) = \rho_K(v_2)$.*

Proof. This is an immediate consequence of the homotopy of v_1 and v_2. In particular, consider the homotopy $F(t, x) = tv_1(x) + (1 - t)\, v_2(x)$, $x \in D$, $0 \leqslant t \leqslant 1$. By the convexity of $V(x)$ for all $x \in D$ it follows that $F(t, \cdot)$ is also a continuous selection of V for all $0 \leqslant t \leqslant 1$. ▮

Lemmas 2.8 and 2.9 justify the following definition of index in multivalued vector fields: Let $V \in C(D \to \Gamma)$ be a multivalued vector field and let K be a Jordan curve in D such that V is regular on K. Then the ***index*** of V is defined as $\rho_K(V) := \rho_K(v)$ where $v \in C(D \to \mathbf{R}^2)$ is any continuous selection of V.

The following lemma is then an immediate consequence of the existence of continuous selections of V on D and of a well-known fact for single valued vector fields (see, e.g., Coddington and Levinson [2], Theorem 4.1, page 398).

(2.10) LEMMA. *Let $V \in C(D \to \Gamma)$ be a vector field and let K be a Jordan curve that $\overline{\mathrm{enc}}(K) \subset D$. If V is regular on $\overline{\mathrm{enc}}(K)$ then $\rho_K(V) = 0$.*

Finally, we shall also need the following

(2.11) LEMMA (Lifshitz [11]). *Let $x \in C([0, T] \to \mathbf{R}^2)$ be a proper parametrization of a Jordan curve K and for $0 < \delta < T$ define $v_\delta(t) := x(t^* + \delta) - x(t)$, $0 \leqslant t \leqslant T$, where $t^* = t$ for $0 \leqslant t \leqslant T - \delta$ and $t^* = t - T$ for $t > T - \delta$. Then $\rho_K(v_\delta) = 1$.*

(2.12) *Proof of Theorem* 2.6. First note that if K has a critical point of V there is nothin to prove and hence assume that V is regular on K. For $\epsilon > 0$ and all $x \in D$ define

$$V_\epsilon(x) := \{y + \epsilon z \mid y \in V(x); \| z \| \leqslant 1\}.$$

Clearly $V_\epsilon \in C(D \to \Gamma)$ and by the continuity of V and the compactness of K, there exists an $\epsilon > 0$ such that V_ϵ is also regular on K. Since $V(x) \subset V_\epsilon(x)$ for all $x \in D$ it is clear that $\rho_K(V_\epsilon) = \rho_K(V)$, and in view of Lemma 2.10 the proof will be complete upon showing that $\rho_K(V_\epsilon) \neq 0$. By Lemma 2.11 this will be accomplished if we can show that for sufficiently small $\delta > 0$, $w_\delta(t) := \delta^{-1}[\varphi(t + \delta) - \varphi(t)] \in V_\epsilon(\varphi(t))$ for all $0 \leqslant t \leqslant T$ (with φ being extended periodically outside the interval $[0, T]$). In view of the uniform continuity of V (and of φ) on K there clearly exists a $\delta > 0$ such that $V(\varphi(\tau)) \subset V_\epsilon(\varphi(t))$ for all $t \in [0, T]$ and all τ such that $| t - \tau | < \delta$. Suppose that $w_\delta(t_0) \notin V_\epsilon(\varphi(t_0))$ for some $t_0 \in [0, T]$. Then there exists a vector $c \neq 0$ and a number α such that $\langle c, w_\delta(t_0)\rangle > \alpha$ and $\langle c, y\rangle \leqslant \alpha$ for all $y \in V_\epsilon(\varphi(t_0))$ (where $\langle\ ,\ \rangle$ denotes inner product).

Since $V(\varphi(t)) \subset V_\epsilon(\varphi(t_0))$ for all $t_0 \leqslant t \leqslant t_0 + \delta$ and since $\dot{\varphi}(t) \in V(\varphi(t))$ a.e., it follows that

$$\langle c, w_\delta(t_0)\rangle = \delta^{-1} \int_{t_0}^{t_0+\delta} \langle c, \dot{\varphi}(\tau)\rangle\, d\tau \leqslant \alpha,$$

a contradiction. This completes the proof. ▮

(2.13) *Outline of Proof of Theorem* 2.5. First note that if L^c has no bounded component then L is a Jordan arc and it is easily seen that every point of L is a critical point of V. Hence, if L is free of critical points, L^c has at least one bounded component. It is also readily noted that the union of some (or all) bounded components of L^c is simply and uniformly locally connected and hence is bounded by a Jordan curve (see Whyburn [12] for definition of uniform local connectedness). The Jordan curve which bounds this set can itself be properly parametrized by a solution of (2.4) and upon application of Theorem 2.6 the result follows. ∎

(2.14) *Remark.* If the vector field V is regular on the Jordan curve K then upon application of Lemma 2.10 to Theorem 2.6 it follows that *every* continuous selection of V has a critical point in enc(K). This fact has interesting implications for control systems as discussed in the next section.

With the aid of Theorem 2.5 we can obtain a much sharper result on existence of critical points in the plane which contains Theorems 2.5 and 2.6 as special cases. First we shall need the following:

(2.15) LEMMA. *Let $V \in C(D \to \Gamma)$ be a multivalued vector field and let $S \subset D$ be compact. Then the set $Q \subset S$ of critical points of V is compact.*

Proof. We only need to show that Q is closed. If not, there exists a limit point x^* of Q which is not in Q and hence $0 \notin V(x^*)$. By the compactness and continuity of $V(x)$ it follows that for each x in some neighborhood of x^*, $0 \notin V(x)$ a contradiction. ∎

(2.16) THEOREM. *Let $V \in C(D \to \Gamma)$ be a multivalued vector field and let $S \subset D$ be compact and connected. If there exists a solution φ of the contingent equation* (2.4) *on the interval* $[0, \infty)$ *whose trajectory is contained in S, then $SCH(S)$ contains a critical point of V.*

Proof. Consider a sequence of positive numbers $\{t_i\}_{i=1}^{\infty}$, $t_i \to \infty$ such that the associated sequence $\{\varphi(t_i)\}_{i=1}^{\infty}$ converges to a limit $x^* \in S$. (Such a sequence exists by virtue of the compactness of S and the fact that $\varphi(\tau) \in S$ for all $\tau \in [0, \infty)$.) Let T_1 and T_2 be any two elements of the sequence $\{t_i\}$ such that $T_2 > T_1 > 0$, and construct the function $\xi \in C([0, T_2] \to \mathbf{R}^2)$ as follows:

$$\xi(t) = \begin{cases} \varphi(t) + \dfrac{t}{T_1}(x^* - \varphi(T_1)) & \text{for } 0 \leqslant t \leqslant T_1, \\ \varphi(t) + \dfrac{t - T_1}{T_2 - T_1}(x^* - \varphi(T_2)) + \dfrac{T_2 - t}{T_2 - T_1}(x^* - \varphi(T_1)) & \\ & \text{for } T_1 < t \leqslant T_2. \end{cases}$$

Clearly this function is absolutely continuous on $[0, T_2]$ and

$$\dot{\xi}(t) = \begin{cases} \dot{\varphi} + \dfrac{1}{T_1}(x^* - \varphi(T_1)) & \text{for } 0 \leqslant t < T_1, \\ \dot{\varphi} + \dfrac{1}{T_2 - T_1}(x^* - \varphi(T_2)) - \dfrac{1}{T_2 - T_1}(x^* - \varphi(T_1)) & \\ & \text{for } T < t < T_2 . \end{cases}$$

Letting

$$\epsilon := \left(\frac{1}{T_1} + \frac{1}{T_2 - T_1}\right) \| x^* - \varphi(T_1)\| + \frac{1}{T_2 - T_1} \| x^* - \varphi(T_2)\|$$

and

$$\delta := \| x^* - \varphi(T_1)\| + \| x^* - \varphi(T_2)\|,$$

we note that

$$\dot{\xi}(t) \in V_\epsilon(\varphi(t)) \qquad \text{a.e. on } [0, T_2]$$

and

$$\xi(t) \in S_\delta \qquad \text{for all} \quad t \in [0, T_2]$$

where $V_\epsilon(\cdot)$ is defined as in the proof of Theorem 2.6, and $S_\delta = \{y + \delta z \mid y \in S, \| z \| \leqslant 1\}$. Obviously, $\xi(t)$ is then a solution on $[0, T_2]$ of the contingent equation $\dot{\xi} \in V_{\epsilon(\delta)}(\xi)$ (where $\epsilon(\delta) > 0$ satisfies the condition that for all $x, y \in S_\delta$, $V_\epsilon(x) \subset V_{\epsilon(\delta)}(y)$ whenever $\| x - y \| \leqslant \delta$). Since clearly $\xi(T_1) = \xi(T_2) = x^*$ it follows by Theorem 2.5 that $V_{\epsilon(\delta)}(\xi)$ must have a critical point in $SCH(S_\delta)$. In view of the convergence of the sequence $\{\varphi(t_i)\}$ to x^* and the continuity of the vector field V, it is clear that by appropriate selection of T_1 and T_2 (to be sufficiently large), the numbers ϵ, δ, and $\epsilon(\delta)$ can be made arbitrarily small. Hence, for some sequence $\{\delta_i\}$, $\delta_i \to 0$, there exists a corresponding sequence $\{\epsilon_i(\delta_i)\}$, $\epsilon_i(\delta_i) \to 0$, such that $V_{\epsilon_i(\delta_i)}$ has a critical point z_i in $SCH(S_{\delta_i})$ for each i. If $z \in SCH(S)$ is the limit of a convergent subsequence of $\{z_i\}$, then in view of Lemma 2.15 z is a critical point of V in $SCH(S)$. ∎

(2.17) COROLLARY. *Assume the conditions of Theorem* 2.16 *hold. If in addition S is simply connected and locally connected then S contains a critical point of V.*

Proof. If S is compact, simply connected and locally connected then $SCH(S) = S$. (This is an immediate consequence of [12, 2.41, p. 34]). ∎

3. REST POINTS OF PLANAR CONTROL SYSTEMS

In this section we shall apply the results of Section 2 to the special case of a planar control system given by Eq. (1.1) with $n = 2$. We shall assume throughout this section that

(3.1) $f(x, u)$ is continuous in both arguments for all $x \in \mathbf{R}^2$ and $u \in \mathbf{R}^m$.

(3.2) For all $x \in \mathbf{R}^2$ the set $f(x, \Omega)$ is compact and convex.

Although assumption (3.1) is in itself not sufficient to insure that for a given admissible control there exists a global solution or even a unique one, the above assumptions are strong enough to carry over the results of Section 2 to the control case. Indeed, if $u \in U_\Omega$ is any admissible control and $\varphi_u(t)$ is some corresponding solution to (1.1), then it is clearly also a solution of the contingent equation

$$\dot{x} \in f(x, \Omega) \tag{3.3}$$

in the sense of Section 2.

Theorem 2.16 can then be restated as follows:

(3.4) THEOREM. *Consider the control system* (1.1) *with* $n = 2$ *and assume that* (3.1) *and* (3.2) *hold. Let* $S \subset \mathbf{R}^2$ *be a compact connected set. If for some admissible control* $u \in U_\Omega$ *there exists a solution* $\varphi_u(t)$ *of* (1.1) *which is contained in* S *for all* $t \in [0, \infty)$ *then* $SCH(S)$ *contains an* Ω*-rest state.*

Corollary 2.17 can be restated for control systems in similar fashion.

(3.5) *Remark.* Theorem 3.4 has the following very interesting and important implication: Under conditions (3.1) and (3.2), if a planar control system has any bounded trajectories, then $\mathbf{R}^2$ contains Ω-rest states. This is a very special situation which may fail already for $n = 3$ as shown in the example of Section 5 below. While the analogous situation is well known for dynamical systems (see, e.g., Jones and Yorke [8]), it is more surprising in the control system setting in that a bounded "controlled" trajectory is a much weaker concept than that of a bounded "free" motion. In fact, the example of Section 5 demonstrates very clearly a situation in which a control system has no rest states *at all* while likewise behaving dynamical systems necessarily have critical points.

An interesting specialization to control systems is obtained by applying Lemma 2.10 to Theorem 2.6 as stated in Remark 2.14

(3.6) THEOREM. *Consider the control system* (1.1) *with* $n = 2$ *and assume* (3.1) *and* (3.2) *hold. Let* K *be a Jordan curve and assume that for an admissible control* $u \in U_\Omega$ *there exists a solution* $\varphi_u(t)$ *which for some* $T > 0$ *is a proper parametrization of* K *on* $[0, T]$. *If* K *has no* Ω*-rest states of* (1.1) *then for each* $\bar{u} \in \Omega$ *there exists* $\bar{x} \in \operatorname{enc}(K)$ *such that* $f(\bar{x}, \bar{u}) = 0$.

Proof. Every constant control provides a continuous selection of $f(\cdot, \Omega)$. ∎

In the case of linear dynamics, i.e., when $f(x, u) = Fx + Gu$ (with F and G constant real matrices), assumption (3.2) (which in particular is satisfied whenever Ω is convex) implies that the set of Ω-rest states is connected. Thus, under the assumption that K has no Ω-rest states, Theorem 3.6 implies that *all* Ω-rest states are enclosed by K. In this case F is necessarily nonsingular since for singular F the set of Ω-rest states is unbounded.

As a final observation in this section it should be noted that assumption (3.2) is not only of *technical* character but is crucial for the existence of Ω-rest states. Specifically, in the absence of convexity of the velocity sets, the existence of bounded trajectories does not generally imply existence of Ω-rest states as seen in the following example which clearly has bounded solutions but no Ω-rest states at all:

$$\dot{x}_1 = \sin u \qquad \Omega = [0, \pi],$$
$$\dot{x}_2 = \cos u.$$

4. Rest Points in Ω-Constrained Sets

In this section we shall consider the control system (1.1) and it will be assumed that the solution $x(t) = \varphi(t, 0, x_0, u)$ of (1.1) arising from the control u and satisfying $x(0) = x_0$ exists and is unique on $[0, \infty)$ for each $u \in U_\Omega$ and each $x_0 \in \mathbf{R}^n$. Specifically we make the following standing assumptions:

(4.1) Ω is a nonempty compact subset of $\mathbf{R}^m$.

(4.2) f is continuous in both arguments and is continuously differentiable in x.

(4.3) The responses of (1.1) are uniformly bounded, i.e., for each $x_0 \in \mathbf{R}^n$ and $T > 0$ there exists $b < \infty$ such that $\| \varphi(t, 0, x_0, u)\| < b$ for all $u \in U_\Omega$ and all $0 \leqslant t \leqslant T$.

(4.4) For all $x \in \mathbf{R}^n$ the set $f(x, \Omega)$ is convex.

(4.5) X is a compact convex subset of $\mathbf{R}^n$ with nonempty interior.

Recall (see, e.g., Lee and Markus [9]) that conditions (4.1)–(4.4) guarantee that the reachable set $F_t(x)$ is compact and depends continuously on t and x for all $t \in [0, \infty)$ and all $x \in \mathbf{R}^n$. Hence the mapping $G: [0, \infty) \times \mathbf{R}^n \to \Gamma$ (Γ being the space of nonempty compact convex subsets of $\mathbf{R}^n$ with Hausdorff topology) defined by $(t, x) \mapsto G(t, x) := H(F_t(x))$ is continuous.

(4.6) Theorem. *Assume* (4.1)–(4.5) *hold and that X is Ω-constrained in time $t^* > 0$. If no point $x \in \partial X$ is t-hull periodic for any $t \in (0, t^*]$, then for each $\bar{u} \in \Omega$ there exists $\bar{x} \in \operatorname{int}(X)$ such that $f(\bar{x}, \bar{u}) = 0$.*

To prove Theorem 4.6 we will first make use of some basic facts about the degree of mappings in $\mathbf{R}^n$ (see, e.g., Dugundji [3]).

(4.7) Lemma. *Let $h \in C(X \to \mathbf{R}^n)$ be a map whose degree $\rho_{\partial X}(h)$ on ∂X is nonzero. Then h is singular on X, i.e., $h(x) = 0$ for some $x \in X$.*

Outline of Proof. Suppose h is regular on X. Then, for $x_0 \in \text{int}(X)$, the map $F(t, x) := h[tx + (1 - t)\, x_0]$, $0 \leqslant t \leqslant 1$, $x \in X$ is a homotopy and it is readily verified that $\rho_{\partial X}(h) = \rho_{\partial X}(F(1, \cdot)) = \rho_{\partial X}(F(0, \cdot)) = 0$ since $F(0, \cdot)$ is a constant map on X. Hence a contradiction. ▮

(4.8) LEMMA. *Let $g \in C(\partial X \to X)$ satisfy $g(x) \neq x$ for all $x \in \partial X$, and define the mapping $h \in C(\partial X \to \mathbf{R}^n)$ by $h(x) = g(x) - x$. Then $\rho_{\partial X}(h) = (-1)^n$.*

Outline of Proof. For $x_0 \in \text{int}(X)$ let $h_0 \in C(\partial X \to \mathbf{R}^n)$ be defined by $h_0(x) = x_0 - x$. Clearly $h(x)$ and $h_0(x)$ are never in opposite direction and hence $\rho_{\partial X}(h) = \rho_{\partial X}(h_0)$. From the definition of degree it is an immediate consequence that $\rho_{\partial X}(h_0) = (-1)^n$ and the result follows. ▮

We shall now, just as in Section 2, extend the definition of degree to multivalued vector fields. Let $W \in C(S \to \Gamma)$ where $S \subset \mathbf{R}^n$. A point $x \in S$ is called a *critical* point of W if $0 \in W(x)$. The vector field W is called *regular* on S if there are no critical points of W in S. As in Lemma 2.9 it is easily seen that if $W \in C(\partial X \to \Gamma)$ is regular on ∂X then $\rho_{\partial X}(w)$ is the same for every continuous selection w of W on ∂X. Hence, we define the *degree* of a regular vector field $W \in C(\partial X \to \Gamma)$ as $\rho_{\partial X}(W) := \rho_{\partial X}(w)$ where w is any continuous selection of W. We also have the following analog of Lemma 2.10:

(4.9) LEMMA. *If $W \in C(X \to \Gamma)$ is regular on X, then $\rho_{\partial X}(W) = 0$.*

Next we shall need the following result:

(4.10) LEMMA. *Let $G \in C(\partial X \to \Gamma)$ satisfy the property that $G(x) \cap X \neq \varnothing$ for all $x \in \partial X$, and define $W \in C(\partial X \to \Gamma)$ by $W(x) = G(x) - x$. If W is regular on ∂X, then $\rho_{\partial X}(W) = (-1)^n$.*

(4.11) *Remark.* If we could make a continuous selection of $G(\cdot) \cap X$, then Lemma 4.10 would be an immediate consequence of Lemma 4.8. However, this may not be possible since the map $x \mapsto G(x) \cap X$ is in general only uppersemicontinuous and not continuous.

(4.12) *Proof of Lemma* 4.10. If $\epsilon > 0$ is sufficiently small, then $W_\epsilon \in C(\partial X \to \Gamma)$ is also regular, where $W_\epsilon(x) = \{x + \epsilon z \mid x \in W(x);\ \| z \| \leqslant 1\}$. Also, since $W(x) \subset W_\epsilon(x)$ for all $x \in \partial X$, it follows that $\rho_{\partial X}(W) = \rho_{\partial X}(W_\epsilon)$. Now the map $x \to G_\epsilon(x) \cap X$ (where $G_\epsilon(x)$ is defined similarly to $W_\epsilon(x)$ is continuous since $G_\epsilon(x) \cap X$ has nonempty interior for each $x \in \partial X$. Applying Lemma 4.8 to a continuous selection g of $G_\epsilon(\cdot) \cap X$ completes the proof. ▮

Finally, we will also make use of the following:

(4.13) LEMMA. *Assume* (4.1)–(4.5) *hold, let $Q(t, x) := H(F_t(x)) - x$ and let $V \in C(X \to \Gamma)$ be defined by $V(x) := f(x, \Omega)$. Then* (i) *V is regular on ∂X if* *and only if there exists $t_1 > 0$ such that $Q(t, \cdot)$ is regular on ∂X for all $0 < t < t_1$.* (ii) *If V is regular, then $\rho_{\partial X}(V) = \rho_{\partial X}(Q(t, \cdot))$ for all $t > 0$ sufficiently small.*

Proof. (i) If V has a critical point $x_0 \in \partial X$ then x_0 is clearly a critical point of $Q(t, \cdot)$ for all $t > 0$. Conversely, suppose that V is regular on ∂X. Let $\epsilon > 0$ be such that V_ϵ is also regular on ∂X, and choose $\delta > 0$ such that $V(y) \subset V_\epsilon(x)$ for all $x, y \in \partial X$ with $\| y - x \| \leqslant \delta$. Then, there exists $t_1 > 0$ such that $y \in Q(t, x)$ implies $\| y \| \leqslant \delta$ for all $x \in \partial X$ and all $t \in (0, t_1]$. We will prove the regularity of $Q(t, \cdot)$ for $t \in (0, t_1)$ by showing that foı each $x \in \partial X$ there exists a hyperplane through the origin such that $Q(t, x)$ is in one of its open half spaces and hence $0 \notin Q(t, x)$. Choose $t \in (0, t_1)$ and $x \in \partial X$. Since $0 \notin V_\epsilon(x)$ there exists a unit vector $c \in \mathbf{R}^n$ such that $\langle c, y \rangle > 0$ for all $y \in V_\epsilon(x)$. If $z \in F_t(x) - x$, there exists a control $u \in U_\Omega$ such that $z = \varphi(t, 0, x, u) - x$. But then $\| \varphi(\tau, 0, x, u) - x \| < \delta$ for all $0 \leqslant \tau \leqslant t$ and hence $\dot\varphi_u(\tau) := d/d\tau[\varphi(\tau, 0, x, u)] \in V_\epsilon(x)$ a.e. on $[0, t]$. Consequently

$$\langle c, z \rangle = \int_0^t \langle c, \dot\varphi_u(\tau) \rangle \, d\tau > 0.$$

Thus, the set $F_t(x) - x$ and also its convex hull $Q(t, x)$ is contained in the open half space $\langle c, z \rangle > 0$. This completes the proof of (i).

(ii) Since for small enough t and each $x \in \partial X$ both $V_\epsilon(x)$ and $Q(t, x)$ are in the same open half space which does not contain the origin (see the construction in the proof of (i)), it follows that the map $P \in C(\partial X \to \Gamma)$ defined by $P(x) := H[V_\epsilon(x) \cup Q(t, x)]$ is regular on ∂X. Hence $\rho_{\partial X}(V) = \rho_{\partial X}(V_\epsilon) = \rho_{\partial X}(P) = \rho_{\partial X}(Q(t, \cdot))$. ∎

(4.14) *Proof of Theorem* 4.6. By the Ω-constrainedness at time t^*, $H(F_{t^*}(x)) \cap X \neq \varnothing$ for each $x \in \partial X$. Hence, by Lemma 4.10, $\rho_{\partial X}(Q(t^*, \cdot)) = (-1)^n$. The nonexistence of t-hull periodic points is equivalent to the regularity of $Q(t, \cdot)$ for all $0 < t \leqslant t^*$. Hence $\rho_{\partial X}(Q(t, \cdot)) = \rho_{\partial X}(Q(t^*, \cdot))$ for all $0 < t \leqslant t^*$. Then by Lemma 4.13, V is regular on ∂X and $\rho_{\partial X}(V) = \rho_{\partial X}(Q(t, \cdot)) = (-1)^n$. For every $\bar u \in \Omega$ the function v defined by $v(x) = f(x, \bar u)$ is a continuous selection of V on X. Hence $\rho_{\partial X}(v) = (-1)^n$ and by Lemma 4.7 v has a critical point in X. ∎

An interesting corollary to Theorem 4.6 is the following

(4.15) Corollary. *Let* (4.1)–(4.5) *hold an assume X is weakly Ω-invariant. Then X contains an Ω-rest state. Moreover, if ∂X contains no Ω-rest states, then for each $\bar u \in \Omega$ there exists $\bar x \in \operatorname{int}(X)$ such that $f(\bar x, \bar u) = 0$.*

Proof. Weak Ω-invariance implies that X is Ω-constrained in time t for each $t > 0$. If ∂X contains no Ω-rest states, then the map V defined in Lemma 4.13 is regular on ∂X and hence, by the same lemma, $Q(t, \cdot)$ is also regular for all t

sufficiently small. Hence ∂X has no t-hull periodic points for small enough t. Theorem 4.6 then holds and the proof is complete. ▮

(4.16) *Remark.* In Feuer and Heymann [4] and Heymann and Stern [5] the existence of Ω-rest states in compact convex weakly Ω-invariant sets was proved using fixed point theorems. The present proof of this fact uses the concept of degree which is in fact much more powerful. In particular, by fixed point arguments one could never conclude that in the absence of boundary rest states there exists an Ω-rest state for *each* $u \in \Omega$.

(4.17) *Remark.* Corollary 4.15 should not be misconstrued to imply, that when ∂X has Ω-rest states then there still necessarily exists a solution x to $f(x, \bar{u}) = 0$ for each $\bar{u} \in \Omega$ (but that the solution may have only "moved out" of X). In fact, in this case, for some $\bar{u} \in \Omega$ the equation $f(x, \bar{u}) = 0$ may be *unsolvable*. This can be readily verified in the example $f(x, u) = x^2 - u$, $\Omega = [-1, 1]$, $X = [-\frac{1}{2}, \frac{1}{2}]$.

(4.18) *Remark.* It is seen from the proof of Theorem 4.6 that the assumption that X is Ω-constrained in time $t^* > 0$ is essential only in that it allows us to conclude that $\rho_{\partial X}(Q(t^*, \cdot)) \neq 0$. This, however, can be assured by various alternative conditions. For example, we could replace Ω-constrainedness by the following more general property:

$$H(F_{t^*}(x)) \cap C(x) \neq \varnothing \qquad \text{for all} \quad x \in \partial X$$

where

$$C(x) = \overline{\{x + \beta(z - x) \mid \beta > 0, z \in X\}}.$$

(4.19) *Remark.* The condition of Ω-constrainedness in itself is far too weak to insure the existence of Ω-rest states. The following simple example illustrates a case wherein (4.1)–(4.5) hold, where for some $t^* > 0$ we have $F_{t^*}(x) \cap \operatorname{int}(X) \neq \varnothing$ for all $x \in X$, where every point of X is t^*-periodic, but where X contains no Ω-rest states. Consider the controlled harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

Here X is the disc of radius 1 centered at (2, 0), Ω is the disc of radius $\frac{1}{2}$ centered at the origin, and $t^* = 2\pi$. As is easily verified, X is Ω-constrained but contains no Ω-rest states since it contains no (nonempty) subset which is weakly Ω-invariant.

A second interesting corollary to Theorem 4.6 is the following result of Jones [7] (see also Jones and Yorke [8]) for existence of critical points in dynamical systems and which we state here in the setting of the present paper. In particular, we shall consider the case where Ω consists of a single point and hence $\varphi(t, 0, x, u) = \varphi(t, 0, x)$, i.e., for each initial state there exists a *unique* solution of (1.1).

(4.20) COROLLARY. *Let Ω consist of a single point and let Y be a homeomorphically convex open subset of $\mathbf{R}^n$. Assume that Y is compactly constrained in time $t^* > 0$ with respect to system* (1.1), *i.e.,* $\varphi(t, 0, x) \in Y$ *for all* $t \geqslant t^*$ *and all* $x \in \bar{Y}$. *Then Y contains an Ω-rest state.*

Proof. Since in this case $F_t(x)$ consists of a single point, the corollary holds for homeomorphically convex sets just as it holds for convex sets. We shall therefore indicate the proof only for the convex case. Note that the compact constrainedness implies Ω-constrainedness and also that no boundary point of Y is t-periodic for any $t > 0$. Hence Theorem 4.6 implies that $\bar{Y}$ contains an Ω-rest state $\bar{x}$. By the uniqueness of solutions, if $\bar{x} \in \partial Y$ then for all time $t > 0$, $\varphi(t, 0, \bar{x}) = \bar{x} \notin Y$. Hence $\bar{x} \in Y$. ∎

(4.21) *Remark.* In Jones [7] the proof of Corollary 4.20 is false in that it relies on a false lemma (Jones [7], Lemma 2) to which counterexamples are easily constructed. While in Jones and Yorke [8] this lemma is not explicitly stated and only an outline of the proof is provided, there is an implicit reliance on the same incorrect assertion.

In the general control case, where Ω does not consist of a single vector, a (weak) analog of compact constrainedness is the property of *strong* Ω-constrainedness. One might then wish to speculate that an analog to Corollary 4.20 holds for strongly Ω-constrained sets in control systems, that is, under this strengthened condition the assumption on the absence of hull periodic solutions becomes superfluous. This is in general false as the example of Section 5 very clearly illustrates. However, in some special cases the control theoretic analog of Corollary 4.20 does hold as is discussed below.

(4.22) THEOREM. *Assume* (4.1)–(4.5) *hold and that f is linear, i.e., $f(x, u) = Fx + Gu$ where F and G are constant real matrices. Assume that X is strongly Ω-constrained. Then X contains an Ω-rest state.*

Proof. Since X is strongly Ω-constrained, it follows that core(X) $\neq \varnothing$ (where core(X) is the largest weakly Ω-invariant subset of X). By the linearity of f (see e.g., Lee and Markus [9] core(X) is compact and convex and hence by Theorem 3.3 of Feuer and Heymann [4] (see also Corollary 4.15 above), core(X) contains an Ω-rest state. ∎

(4.23) THEOREM. *Assume* (4.1)–(4.5) *hold and that $n \leqslant 2$ (where n is the dimension of the state space) and assume that X is strongly Ω-constrained. Then X contains an Ω-rest state.*

Proof. Note as in the proof of Theorem 4.22 that core(X) $\neq \varnothing$, and hence in the case $n = 2$ since X is compact and simply connected the theorem follows from Theorem 3.4 and the fact that X is also locally connected (see also Corollary 2.17). Since the case $n = 1$ can be embedded in a two dimensional space the result clearly follows then too. ∎

5. The Nonexistence of Ω-Rest States—An Example

In the present section we demonstrate via an example two important facts which are related to the results of the present paper: (1) that the existence of bounded controlled trajectories in control systems of dimension higher than two does not insure the existence of Ω-rest states in the state space even when $f(x, \Omega)$ is convex for all $x \in \mathbf{R}^n$; and (2) that strongly Ω-constrained sets in $\mathbf{R}^n$ do not necessarily possess Ω-rest states when $n \geqslant 3$. We shall build up our example as we go along so as to enhance the intuitive insight of the reader.

We consider first the planar control system given by

$$\begin{aligned} \dot{x} &= ux - y, \\ \dot{y} &= x + uy, \end{aligned} \qquad \Omega = \{u \mid |u| \leqslant 1\}, \tag{5.1}$$

where $n = 2$ and $m = 1$. Upon substitution of $x = r\cos\varphi$ and $y = r\sin\varphi$, we obtain the equations

$$\begin{aligned} \dot{r} &= ur, \\ \dot{\varphi} &= 1, \end{aligned} \tag{5.2}$$

which are valid for all (x, y) so long as $r^2 = x^2 + y^2 \neq 0$. Hence, the only Ω-rest state for this system is the origin which satisfies $\dot{y} = \dot{x} = 0$ for all $u \in \Omega$. For each fixed u, the motion is orbital (around the origin) with constant angular velocity which is independent of the radius. Also, the control available permits transition between orbits (although the origin can be reached only asymptotically). Consider now the three dimensional system

$$\begin{aligned} \dot{x} &= ux - y, \\ \dot{y} &= x + uy, \\ \dot{z} &= 1 - x^2 - y^2. \end{aligned} \qquad \Omega = \{u \mid |u| \leqslant 1\}, \tag{5.3}$$

This system has clearly *no* Ω-rest states at all. Indeed, $\dot{z} = 1$ for any point $(x, y, z) = (0, 0, z)$, and for any point away from the z-axis the polar equations

$$\begin{aligned} \dot{r} &= ur, \\ \dot{\varphi} &= 1, \\ \dot{z} &= 1 - r^2, \end{aligned} \tag{5.4}$$

are valid and $\dot{\varphi} \neq 0$ for all choices of r and u. Consider now the cylinder

$$C = \{(x, y, z) \mid x^2 + y^2 \leqslant 2; |z| \leqslant 1\}.$$

First observe that any initial point in C for which $x^2 + y^2 = r^2 = 1$ is kept in C using the control $u = 0$. Indeed, the resulting motion is circular with radius

$r = 1$. Since $\dot{z} = 0$ along such a motion, it follows that $z(t) = z(0)$ for all $t > 0$. Hence (5.3) shows that the mere presence of bounded controlled trajectories does not insure the existence of Ω-rest states in the space.

Using elementary control theoretic considerations, it can be seen that for each point in C which is *not* on the z-axis (i.e., $r \neq 0$), there exists a control which drives the trajectory into the surface $\{x^2 + y^2 = 1, \mid z \mid < 1\}$ in some finite time $T > 0$. Yet, if the initial point is on the z-axis, then $r = 0$ and $\dot{z} = 1$ along any controlled motion. Hence, such a motion will escape to infinity along the z-axis regardless of the control. To rectify this last prevailing difficulty we consider the modified system:

$$\begin{aligned} \dot{x} &= ux - y + v, \\ \dot{y} &= x + uy + w, \qquad \Omega = \{(u, v, w) \mid \mid u \mid \leqslant 1; \mid v \mid \leqslant v_0; \mid w \mid \leqslant w_0\}, \\ \dot{z} &= 1 - x^2 - y^2, \end{aligned} \tag{5.5}$$

where $v_0 > 0$ and $w_0 > 0$. In this case it can be seen that C is strongly Ω-constrained. Yet, it is easily verified that if $v_0 < \frac{1}{2}$ and $w_0 < \frac{1}{2}$, the system (5.5) still has no Ω-rest states in $\mathbf{R}^3$ (and certainly none in C).

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